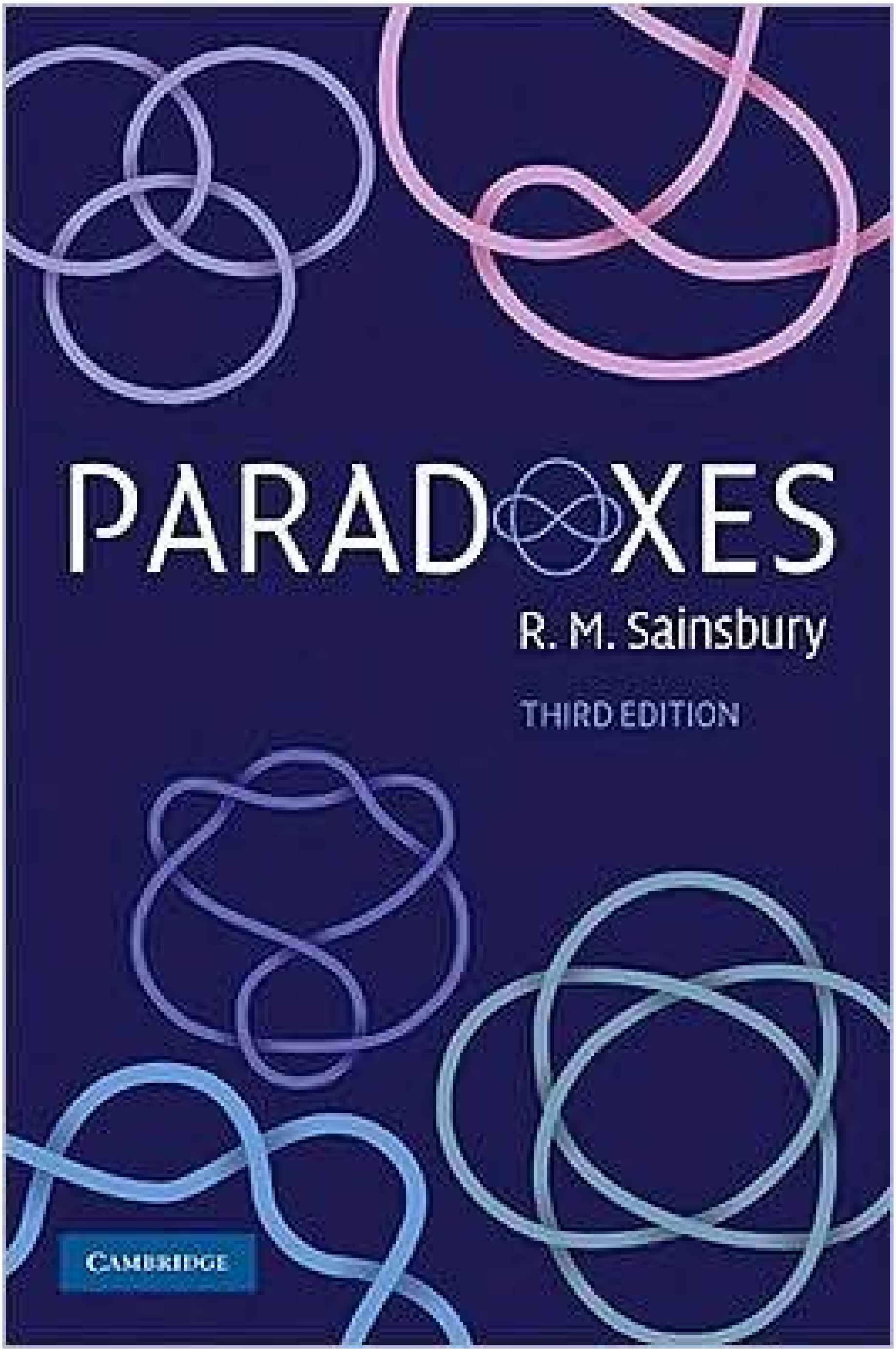
Paradoxes



Paradoxes

THIRD EDITION

R. M. Sainsbury

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Foreword to third edition

The main change in this edition is the addition of a new chapter on moral paradoxes, which I was inspired to write by reading Smilansky's excellent book *Ten Moral Paradoxes*. To Saul Smilansky I owe thanks for encouragement and comments. The new chapter is numbered 2 and subsequent chapters are renumbered accordingly. I placed the new chapter early in the book in the belief that the discussion is more straightforward than just about any of the other chapters.

I have made some small changes elsewhere, notably to the chapter on vagueness (now chapter 3) and to the suggested reading. Since the second edition appeared in 1995, the internet has transformed many aspects of our life. There are now many websites which help people doing philosophy at every level. The *Stanford Encyclopedia of Philosophy* (plato.stanford.edu) should be the first place to turn if your curiosity has been aroused by this text. Someone with internet access can now do serious research in philosophy, even without the advantage of a university library.

My thanks to Daniel Hill for many useful suggestions for this edition.

Introduction

Paradoxes are fun. In most cases, they are easy to state and immediately provoke one into trying to "solve" them.

One of the hardest paradoxes to handle is also one of the easiest to state: the Liar paradox. One version of it asks you to consider the man who simply says, "What I am now saying is false." Is what he says true or false? The problem is that if he speaks truly, he is truly saying that what he says is false, so he is speaking falsely; but if he is speaking falsely, then, since this is just what he says he is doing, he must be speaking truly. So if what he says is false, it is true; and if it is true, it is false. This paradox is said to have "tormented many ancient logicians and caused the premature death of at least one of them, Philetas of Cos." Fun can go too far.

Paradoxes are serious. Unlike party puzzles and teasers, which are also fun, paradoxes raise serious problems. Historically, they are associated with crises in thought and with revolutionary advances. To grapple with them is not merely to engage in an intellectual game, but is to come to grips with key issues. In this book, I report some famous (and some less famous) paradoxes and indicate how one might respond to them. These responses lead into some rather deep waters.

This is what I understand by a paradox: an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises. Appearances have to deceive, since the acceptable cannot lead by acceptable steps to the unacceptable. So, generally, we have a choice: either the conclusion is not really unacceptable, or else the starting point, or the reasoning, has some non-obvious flaw.

Paradoxes come in degrees, depending on how well appearance camouflages reality. Let us pretend that we can represent how paradoxical something is on a ten-point scale. The weak or shallow end we shall label 1; the cataclysmic end, home of paradoxes that send seismic shudders through a wide region of thought, we shall label 10. Serving as a marker for the point labeled 1 is the so-called Barber paradox: in a certain remote Sicilian village, approached by a long ascent up a precipitous mountain road, the barber shaves all and only those villagers who do not shave themselves. Who shaves the barber? If he himself does, then he does not (since he shaves *only* those who do not shave themselves); if he does not, then he indeed does (since he shaves *all* those who do not shave themselves). The unacceptable supposition is that there is such a barber – one who shaves himself if and only if he does not. The story may have sounded acceptable: it turned our minds, agreeably enough, to the mountains of inland Sicily. However, once we see what the consequences are, we realize that the story cannot be true: there cannot be such a barber, or such a village. The story is unacceptable. This is not a very deep paradox because the unacceptability is very thinly disguised by the mountains and the remoteness.

At the other end of the scale, the point labeled 10, I shall place the Liar. This placing seems the least that is owed to the memory of Philetas.

The deeper the paradox, the more controversial is the question of how one should respond to it. Almost all the paradoxes I discuss in the ensuing chapters score 6 or higher on the scale, so they are really serious. (Some of those in chapter 2 and in appendix I might be argued to rate a lower score.) This means that there is severe and unresolved disagreement about how one should deal with them. In many cases, though certainly not all (not, for example, in the case of the Liar), I have a definite view; but I must emphasize that, although I naturally think my own view is correct, other and greater men have held views that are diametrically opposed. To get a feel for how controversial some of the issues are, I suggest examining the suggestions for further reading at the ends of chapters.

Some paradoxes collect naturally into groups by subject matter. The paradoxes of Zeno which I discuss form a group because they all deal with space, time, and infinity. The paradoxes of chapter 4 form a group because they bear upon the notion of rational action. Some groupings are controversial. For example, Russell grouped the paradox about classes with the Liar paradox. In the 1920s, Ramsey argued that this grouping disguised a major difference. More recently, it has been argued that Russell was closer to the truth than Ramsey.

I have compared some of the paradoxes treated within a single chapter, but I have made no attempt to portray larger patterns. However, it is arguable that there are such patterns, or even that the many paradoxes are the many signs of one "master cognitive flaw." This last claim has been ingeniously argued by Roy Sorensen (1988).

Questions can be found in boxes throughout the text. I hope that considering these will give pleasure and will prompt the reader to elaborate some of the themes in the text. Asterisked questions are referred to in appendix II, where I have made a point that might be relevant to an answer.

I feel that chapter 6 is the hardest; it might well be left until last. The first and second are probably the easiest. The order of the others is arbitrary. Chapter 7 does not introduce a paradox, but rather examines the assumption, made in the earlier chapters, that all contradictions are unacceptable. I think it would not make much sense to one completely unfamiliar with the topics discussed in chapter 6.

I face a dilemma: I find a book disappointing if the author does not express his own beliefs. What holds him back from stating, and arguing for, the truth as he sees it? I could not bring myself to exercise this restraint. On the other hand, I certainly would not want anyone to believe what I say without first carefully considering the alternatives. So I must offer somewhat paradoxical advice: be very skeptical about the proposed "solutions"; they are, I believe, correct.

Suggested reading

There are now a number of excellent books that deal with a spectrum of paradoxes, in particular Nicholas Rescher (2001) *Paradoxes: Their Roots, Range and Resolution* and Roy Sorensen (2003) *A Brief History of the Paradox: Philosophy and the Labyrinths of the Mind*. There is also a surprisingly large amount of material on the web. The following webpage lists a whole range of paradox sites, of very diverse kinds: www.google.com/Top/Society/Philosophy/Philosophy_of_Logic/Paradoxes/.

1 Zeno's paradoxes: space, time, and motion

1.1 Introduction

Zeno the Greek lived in Elea (a town in what is now southern Italy) in the fifth century BC. The paradox for which he is best known today concerns the great warrior Achilles and a previously unknown tortoise. For some reason now lost in the folds of time, a race was arranged between them. Since Achilles could run much faster than the tortoise, the tortoise was given a head start. Zeno's astonishing contribution is a "proof" that Achilles could never catch up with the tortoise no matter how fast he ran and no matter how long the race went on.

The supposed proof goes like this. The first thing Achilles has to do is to get to the place from which the tortoise started. The tortoise, although slow, is unflagging: while Achilles is occupied in making up his handicap, the tortoise advances a little bit further. So the next thing Achilles has to do is to get to the *new* place the tortoise occupies. While he is doing this, the tortoise will have gone on a little bit further still. However small the gap that remains, it will take Achilles some time to cross it, and in that time the tortoise will have created another gap. So however fast Achilles runs, all the tortoise need do in order not to be beaten is keep going – to make *some* progress in the time it takes Achilles to close the previous gap between them.

No one nowadays would dream of accepting the conclusion that Achilles cannot catch the tortoise. (I will not vouch for Zeno's reaction to his paradox: sometimes he is reported as having taken his paradoxical conclusions quite seriously and literally, showing that motion was impossible.) Therefore, there must be something wrong with the argument. Saying exactly *what* is wrong is not easy, and there is no uncontroversial diagnosis. Some have seen the paradox as produced by the assumption that space or time is infinitely divisible, and thus as genuinely proving that space or time is *not* infinitely divisible. Others have seen in the argument nothing more than a display of ignorance of elementary mathematics – an ignorance perhaps excusable in Zeno's time but inexcusable today.

The paradox of Achilles and the tortoise is Zeno's most famous, but there were several others. The Achilles paradox takes for granted that Achilles can start running, and purports to prove that he cannot get as far as we all know he can. This paradox dovetails nicely with one known as the Racetrack, or Dichotomy, which purports to show that nothing can *begin* to move. In order to get anywhere, say to a point one foot ahead of you, you must first get halfway there. To get to the halfway point, you must first get halfway to *that* point. In short, in order to get anywhere, even to begin to move, you must first perform an infinity of other movements. Since this seems impossible, it seems impossible that anything should move at all.

Almost none of Zeno's work survives as such. For the most part, our knowledge of what his arguments were is derived from reports by other philosophers, notably Aristotle. He presents Zeno's arguments very briefly, no doubt in the expectation that they would be familiar to his audience from the oral tradition that was perhaps his own only source. Aristotle's accounts are so compressed that only by guesswork can one reconstruct a detailed argument. The upshot is that there is no universal agreement about what should count as "Zeno's paradoxes," or about exactly what his arguments were. I shall select arguments that I believe to be interesting and important, and which are commonly attributed to Zeno, but I make no claim to be expounding what the real, historical Zeno actually said or thought.

Aristotle is an example of a great thinker who believed that Zeno was to be taken seriously and not dismissed as a mere propounder of childish riddles. By contrast, Charles Peirce wrote of the Achilles paradox: "this ridiculous little catch presents no difficulty at all to a mind adequately trained in mathematics and in logic, but is one of those which is very apt to excite minds of a certain class to an obstinate determination to believe a given proposition" (1935, vol. VI, §177, p. 122). On balance, history has sided with Aristotle, whose view on this point has been shared by thinkers as dissimilar as Hegel and Russell.

I shall discuss three Zenonian paradoxes concerning motion: the Racetrack, the Achilles, and a paradox known as the Arrow. Before doing so, however, it will be useful to consider yet another of Zeno's paradoxes, one that concerns space. Sorting out this paradox provides the groundwork for tackling the paradoxes of motion.

1.2 Space

In ancient times, a frequently discussed perplexity was how something ("one and the same thing") could be both one and many. For example, a book is one but also many (words or pages); likewise, a tree is one but also many (leaves, branches, molecules, or whatever). Nowadays, this is unlikely to strike anyone as very problematic. When we say that the book or the tree *is* many things, we do not mean that it is identical with many things (which would be absurd), but rather that it is made up of many parts. Furthermore, at least on the face of it, there is nothing especially problematic about this relationship between a whole and the parts which compose it (see question 1.1).

1.1

Appearances may deceive. Let us call some particular tree *T*, and the collection of its parts at a particular moment *P*. Since trees can survive the loss of some of their parts (e.g. their leaves in the fall), *T* can exist when *P* no longer does. Does this mean that *T* is something other than *P* or, more generally, that each thing is distinct from the sum of its parts? Can *P* exist when *T* does not (e.g. if the parts of the tree are dispersed by timber-felling operations)?

Zeno, like his teacher Parmenides, wished to argue that in such cases there are not many things but only one thing. I shall examine one ingredient of this argument. Consider any region of space, for example the region occupied by this book. The region can be thought of as having parts which are themselves spatial, that is, they have some size. This holds however small we make the parts. Hence, the argument runs, no region of space is "infinitely divisible" in the sense of containing an *infinite* number of spatial parts. For each part has a size, and a region composed of an infinite number of parts of this size must be infinite in size.

This argument played the following role in Zeno's attempt to show that it is not the case that there are "many things." He was talking only of objects in space, and he assumed that an object has a part corresponding to every part of the space it fills. He claimed to show that, if you allow that objects have parts at all, you must say that each object is infinitely large, which is absurd. You must therefore deny that objects have parts. From this Zeno went on to argue that *plurality* – the existence of many things – was impossible. I shall not consider this further development, but will instead return to the argument against infinite divisibility upon which it draws (see question 1.2).

1.2

* Given as a premise that no object has parts, how could one attempt to argue that there is no more than one object?

The conclusion may seem surprising. Surely one could convince oneself that any space has infinitely many spatial parts. Suppose we take a rectangle and bisect it vertically to give two further rectangles. Taking the right-hand one, bisect it vertically to give two more new rectangles. Cannot this process of bisection go on indefinitely, at least in theory? If so, any spatial area is made up of infinitely many others.

Wait one moment! Suppose that I am drawing the bisections with a ruler and pencil. However thin the pencil, the time will fairly soon come when, instead of producing fresh rectangles, the new lines will fuse into a smudge. Alternatively, suppose that I am cutting the rectangles from paper with scissors. Again, the time will fairly soon come when my strip of paper will be too small to cut. More scientifically, such a process of physical division must presumably come to an end *sometime*: at the very latest, when the remainder of the object is no wider than an atom (proton, hadron, quark, or whatever).

The proponent of infinite divisibility must claim to have no such physical process in mind, but rather to be presenting a purely intellectual process: for every rectangle we can consider, we can also consider a smaller one having half the width. This is how we conceive any space, regardless of its shape. What we have to discuss, therefore, is whether the earlier argument demonstrates that space cannot be as we tend to conceive it; whether, that is, the earlier argument succeeded in showing that no region could have infinitely many parts.

We all know that there are finite spaces which have spatial parts, but the argument supposedly shows that there are not. Therefore we must reject one of the premises that leads to this absurd conclusion, and the most suitable for rejection, because it is the most controversial, is that space is infinitely divisible. This premise supposedly forces us to say that either the parts of a supposedly infinitely divisible space are finite in size, or they are not. If the latter holds, then they are nothing, and no number of them could together compose a finite space. If the former holds, infinitely many of them together will compose an infinitely large space. Either way, on the supposition that space is infinitely divisible, there are no finite spaces. Since there obviously are finite spaces, the supposition must be rejected.

The notion of infinite divisibility remains ambiguous. On the one hand, to say that any space is infinitely divisible could mean that there is no upper limit to the number of imaginary operations of dividing we could effect. On the other hand, it could mean that the space contains an infinite number of parts. It is not obvious that the latter follows from the former. The latter claim might seem to rely on the idea that the process of imaginary dividings could somehow be "completed." For the moment let us assume that the thesis of infinite divisibility at stake is the thesis that space contains infinitely many non-overlapping parts, and that each part has some finite size.

The most doubtful part of the argument against the thesis is the claim that a space composed of an infinity of parts, each finite in size, must be infinite. This claim is incorrect, and one way to show it is to appeal to mathematics. Let us represent the imagined successive bisections by the following series:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots$$

where the first term $\left(\frac{1}{2}\right)$ represents the fact that, after the first bisection, the right-hand rectangle is only half the area of the original rectangle; and similarly for the other terms. Every member of this series is a finite number, just as each of the spatial parts is of finite size. This does not mean that the sum of the series is infinite. On the contrary, mathematics texts have it that this series sums to 1. If we find nothing problematic in the idea that an infinite collection of finite numbers has a finite sum, then by analogy we should be happy with the idea that an infinite collection of finite spatial parts can compose a finite spatial region (see question 1.3).

This argument from mathematics establishes the analogous point about space (namely, that infinitely many parts of finite size may together form a finite whole) only upon the assumption that the analogy is good: that space, in the respect in question, has the properties that numbers have. This is controversial. For example, we have already said that some people take Zeno's paradoxes to show that space is not continuous, although the series of numbers is. Hence we would do well to approach the issue again. We do not have to rely on any mathematical argument to show that a finite whole can be composed of an infinite number of finite parts.

There are two rather similar propositions, one true and one false, and we must be careful not to confuse them.

(1) If, for some finite size, a whole contains infinitely many parts none smaller than this size, then the whole is infinitely large.
(2) If a whole contains infinitely many parts, each of some finite size, then the whole is infinitely large.

Statement (1) is true. To see this, let the minimum size of the parts be δ (say linear or square or cubic inches). Then the size of the whole is $\infty \times \delta$, which is clearly an infinite number. However, (1) does not bear on the case we are considering. To see this, let us revert to our imagined bisections. The idea was that however small the remaining area was, we could

1.3

Someone might object: is it not just a *convention* in mathematics to treat this series as summing to 1? More generally, is it not just a convention to treat the sum of an infinite series as the limit of the partial sums? If this is a mere mathematical convention, how can it tell us anything about space? Readers with mathematical backgrounds might like to comment on the following argument, which purports to show that the fact that the series sums to 1 can be derived from ordinary arithmetical notions, without appeal to any special convention. (*Warning*: mathematicians tell me that what follows is highly suspect!)

The series can be represented as

$$x + x^2 + x^3 + \cdots$$

where $x = ½$. Multiplying this expression by x has the effect of lopping off the first term:

$$x(x + x^2 + x^3 + \cdots) = x^2 + x^3 + x^4 + \cdots$$

Here we apply a generalization of the principle of distribution:

$$a.(b + c) = (a.b) + (a.c).$$

Using this together with a similar generalization of the principle that

$$(1 - a).(b + c) = (b + c) - a.(b + c)$$

we get:

$$(1 - x).(x + x^2 + x^3 + \cdots) = (x + x^2 + x^3 + \cdots) - (x^2 + x^3 + x^4 + \cdots)$$

Thus

$$(1 - x).(x + x^2 + x^3 + x \ldots) = x$$

So, dividing both sides by $(1 - x)$:

$$x + x^2 + x^3 + \cdots = \frac{x}{(1 - x)}$$

So where $x = ½$, the sum of the series is equal to 1.

always imagine it being divided into two. This means that there can be no lower limit on how small the parts are. There can be no size δ such that all the parts are at least this big. For any such size, we can always imagine it being divided into two.

To see that (2) is false, we need to remember that it is essential to the idea of infinite divisibility that the parts get smaller, without limit, as the imagined process of division proceeds. This gives us an almost visual way of understanding how the endless series of rectangles can fit into the original rectangle: by getting progressively smaller.

It would be as wrong to infer "There is a finite size which every part possesses" from "Every part has some finite size or other" as it would be to infer "There is a woman who is loved by every man" from "Every man loves some woman or other." (Readers trained in formal logic will recognize a quantifier-shift fallacy here: one cannot infer an ∃∀ conclusion from the corresponding ∀∃ premise.)

The explanation for any tendency to believe that (2) is true lies in a tendency to confuse it with (1). We perhaps tend to think: *at the end of the series* the *last* pair of rectangles formed have some finite size, and all the other infinitely many rectangles are larger. Therefore, taken together they must make up an infinite area. However, there is *no such thing* as the last pair of rectangles to be formed: our infinite series of divisions has no last member. Once we hold clearly in mind that there can be no lower limit on the size of the parts induced by the infinite series of envisaged divisions, there is no inclination to suppose that having infinitely many parts entails being infinitely large.

The upshot is that there is no contradiction in the idea that space is infinitely divisible, in the sense of being composed of infinitely many non-overlapping spatial parts, each of some finite (non-zero) size. This does not establish that space *is* infinitely divisible. Perhaps it is granular, in the way in which, according to quantum theory, energy is. Perhaps there are small spatial regions that have no distinct subregions. The present point, however, is that the Zenonian argument we have discussed gives us no reason at all to believe this granular hypothesis.

This supposed paradox about space may well not strike us as very deep, especially if we have some familiarity with the currently orthodox mathematical treatment of infinity. Still, we must not forget that current orthodoxy was not developed without a struggle, and was achieved several centuries after Zeno had pondered these questions. Zeno and his contemporaries might with good reason have had more trouble with it than we do. The position of a paradox on the ten-point scale mentioned in the introduction can change over time: as we become more sophisticated detectors of mere appearance, a paradox can slide down toward the Barber end of the scale.

Clearing this paradox out of the way will prove to have been an essential preliminary to discussing Zeno's deeper paradoxes, which concern motion.

1.3 The Racetrack

If a runner is to reach the end of the track, he must first complete an infinite number of different journeys: getting to the midpoint, then to the point midway between the midpoint and the end, then to the point midway between this one and the end, and so on. Since it is logically impossible for someone to complete an infinite series of journeys, the runner cannot reach the end of the track. It is irrelevant how far away the end of the track is – it could be just a few inches away – so this argument, if sound, will show that all motion is impossible. Moving to any point will involve an infinite number of journeys, and an infinite number of journeys cannot be completed in a finite time.

Let us call the starting point Z (for Zeno), and the endpoint $Z^\star$. The argument can be analyzed into two premises and a conclusion, as follows:

(1) Going from Z to $Z^\star$ would require one to complete an infinite number of journeys: from Z to the point midway to $Z^\star$, call it Z_1; from Z_1 to the point midway between it and $Z^\star$, call it Z_2; and so on.
(2) It is logically impossible for anyone (or anything) to complete an infinite number of journeys.

Conclusion: It is logically impossible for anyone to go from Z to $Z^\star$. Since these points are arbitrary, *all* motion is impossible.

Apparently acceptable premises, (1) and (2), lead by apparently acceptable reasoning to an apparently unacceptable conclusion.

No one nowadays would for a moment entertain the idea that the conclusion is, despite appearances, acceptable. (I refrain from vouching for Zeno's own response.) Moreover, the reasoning appears impeccable. So for us the question is this: which premise is incorrect, and why?

Let us begin by considering premise (1). The idea is that we can generate an infinite series, let us call it the Z-series, whose terms are

$$Z, Z_1, Z_2, \ldots$$

These terms, it is proposed, can be used to analyze the journey from Z to $Z^\star$, for they are among the points that a runner from Z to $Z^\star$ must pass through en route. However, $Z^\star$ itself is not a term in the series; that is, it is not generated by the operation that generates new terms in the series – halving the distance that remains between the previous term and $Z^\star$.

The word "journey" has, in the context, some misleading implications. Perhaps "journey" connotes an event done with certain intentions, but it is obvious that a runner could form no intention with respect to most of the members of the Z-series, for he would have neither the time, nor the memory, nor the conceptual apparatus to think about most of them. Furthermore, he may well form no intention with respect to those he *can* think about. Still, if we explicitly set these connotations aside, then (1) seems hard to deny, once the infinite divisibility of space is granted; for then all (1) means is the apparent platitude that motion from Z to $Z^\star$ involves traversing the distances Z to Z_1, Z_1 to Z_2, and so on.

Suspicion focuses on (2). Why should one not be able to complete an infinite number of journeys in a finite time? Is that not precisely what *does* happen when anything moves? Furthermore, is it not something that *could* happen even in other cases? For example, consider a view that Bertrand Russell once affirmed: he argued that we could imagine someone getting more and more skillful in performing a given task, and so completing it more and more quickly. On the first occasion, it might take one minute to do the job, on the second, only a half a minute, and so on, so that, performing the tasks consecutively, the whole series of infinitely many could be performed in the space of two minutes. Russell said, indeed, that this was "medically impossible" but he held that it was *logically* possible: no contradiction was involved. If Russell is right about this, then (2) is the premise we should reject.

However, consider the following argument, in which the word "task" is used in quite a general way, so as to subsume what we have been calling "journeys."

> There are certain reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off.
>
> Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half-minute, and so on, according to Russell's recipe. After I have completed the whole infinite sequence of jabs, i.e., at the end of two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction. (Thomson 1954; cited in Gale 1968, p. 411)

Let us call the envisaged setup consisting of me, the switch, the lamp, and so on, "Thomson's lamp." The argument purports to show that Thomson's lamp cannot complete an infinite series of switchings in a finite time. It proceeds by *reductio ad absurdum*: we suppose that it *can* complete such a series, and show that this supposition leads to an absurdity – that the lamp is neither on nor off at the supposed end of the series of tasks.

The argument is not valid. The supposition that the infinite series has been completed does not lead to the absurdity that the lamp is neither on nor off. Nothing follows from this supposition about the state of the lamp *after* the infinite series of switchings.

Consider the series of moments T_1, T_2, …, each corresponding to a switching. According to the story, the gaps between the members of this T-series get smaller and smaller, and the rate of switching increases. At T_1 a switching on occurs, at T_2 a switching off occurs, and so on. Call the first moment after the (supposed) completion of the series $T^\star$. It follows from the specification of the infinite series that, for any moment *in the T-series*, if the lamp is on at that moment there is a later moment in the series at which the lamp is off; and vice versa. However, nothing follows from this about whether the lamp is on or off at $T^\star$, for $T^\star$ does *not belong* to the T-series. $T^\star$ is not generated by the operation that generates new members of the T-series from old, being a time half as remote from the old member as its predecessor was from it. The specification of the task speaks only to members of the T-series, and this has no consequences, let alone contradictory consequences, for how things are at $T^\star$, which lies outside the series (see question 1.4).

1.4

Are we entitled to speak of "the first moment after the (supposed) completion of the task"?

The preceding paragraph is not designed to prove that it is logically possible for an infinite series of tasks to be completed. It is designed to show only that Thomson's argument against this possibility fails. In fact, someone might suggest a reason of a different kind for thinking that there is a logical absurdity in the idea of Thomson's lamp.

Consider the lamp's button. We imagine it to move the same distance for each switching. If it has moved infinitely many times, then an infinite distance has been traversed at a finite speed in a finite time. There is a case for saying that this is logically impossible, for there is a case for saying that what we *mean* by average speed is simply distance divided by total time. It follows that if speed and total time are finite, so is distance. If this is allowed, then Thomson was right to say that Thomson's lamp as he described it is a logical impossibility, even though the argument he gave for this conclusion was unsatisfactory.

This objection might be countered by varying the design of the machine. There are at least two possibilities. One is that the machine's switch be so constructed that if on its first switching it travels through a distance δ, then on the second switching it travels $\delta/2$, on the third $\delta/4$, and so on. Another is that the switch be so constructed that it travels faster and faster on each switching, without limit (see questions 1.5, 1.6).

1.5

Does this mean that it would have to travel infinitely fast in the end?

1.6

* Does this mean that the switch would have to travel faster than the speed of light? If so, does this mean that the machine is *logically* impossible?

It is hard to find positive arguments for the conclusion that this machine is logically possible; but this machine is open neither to Thomson's objection, which was invalid, nor to the objection that it involves an infinite distance being traveled in a finite time. Therefore, until some other objection is forthcoming, we can (provisionally, and with due caution) accept this revised Thomson's lamp as a logical possibility. What is more, if *it* is a possibility, then there is nothing logically impossible about a runner completing an infinite series of journeys (see question 1.7).

1.7

Evaluate the following argument:

> We can all agree that the series of numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ sums to 1. What is controversial is whether this fact has any bearing on whether the runner can reach $Z^\star$. We know that it would be absurd to say that energy is infinitely divisible merely because for any number that is used to measure some quantity of energy there is a smaller one. Likewise, Zeno's paradox of the runner shows that motion through space should not be thought of as an endless progression through an infinite series. It is as clear that there is a smallest motion a runner can make as it is that there is a smallest spatial distance that we are capable of measuring.

One does not need to establish outré possibilities, such as that of a Thomson's lamp that can complete an infinite number of tasks, in order to establish that the runner can reach $Z^\star$. The argument is supposed to work the other way: if even the infinite Thomson's lamp is possible, then there can be no problem about the runner.

In the next section, I discuss a rather sophisticated variant of the Racetrack. The discussion may help resolve some of the worries that remain with this paradox.

1.4 The Racetrack again

Premise (1) of the previous section asserted that a necessary condition for moving from Z to $Z^\star$ is moving through the infinite series of intermediate Z-points. In this rerun, I want to consider a different problem. It is that there appear to be persuasive arguments for the following inconsistent conclusions:

(a) Passing through all the Z-points is sufficient for reaching $Z^\star$.
(b) Passing through all the Z-points is *not* sufficient for reaching $Z^\star$.

We cannot accept both (a) and (b). The contradiction might be used to disprove the view that the runner's journey can be analyzed in terms of an infinite series, and this would throw doubt on our earlier premise (1) (p. 11).

Let us look more closely at an argument for (a):

> Suppose someone could have occupied every point in the Z-series without having occupied any point outside it, in particular without having occupied $Z^\star$. Where would he be? Not at any Z-point, for then there would be an unoccupied Z-point to the right. Not, for the same reason, between Z-points. And, *ex hypothesi*, not at any point external to the Z-series. But these possibilities are exhaustive. (Cf. Thomson 1954; cited in Gale 1968, p. 418)

In other words, if you pass through all the Z-points, you *must* get to $Z^\star$. Contrasted with this is a simple argument against sufficiency – an argument for (b):

$Z^\star$ lies outside the Z-series. It is further to the right than any member of the Z-series. So going through all the members of the Z-series cannot take you as far to the right as $Z^\star$. So reaching $Z^\star$ is not logically entailed by passing through every Z-point.

The new twist to the Racetrack is that we have plausible arguments for both (a) and (b), but these are inconsistent.

The following objection to the argument for (a) has been proposed by Paul Benacerraf (1962, p. 774). A possible answer to the question "Where would the runner be after passing through all the Z-points?" is "Nowhere!" Passing through all the Z-points is not sufficient for arriving at $Z^\star$ because one might cease to exist after reaching every Z-point but without reaching $Z^\star$. To lend color to this suggestion, Benacerraf invites us to imagine a genie who "shrinks from the thought" of reaching $Z^\star$ to such an extent that he gets progressively smaller as his journey progresses. By Z_1 he is half his original size, by Z_2 a quarter of it, and so on. Thus by the time he has passed through every Z-point his size is zero, and "there is not enough left of him" to occupy $Z^\star$.

Even if this is accepted (see question 1.8), it will not resolve our problem. The most that it could achieve is a qualification of (a). What would have to be said to be sufficient for reaching $Z^\star$ is not merely passing through every Z-point, but doing that and *also* (!) continuing to exist. However, the argument against sufficiency, if it is good at all, seems just as good against a correspondingly modified version of (b). Since $Z^\star$ lies outside the Z-series, even passing through every Z-point *and* continuing to exist cannot logically guarantee arriving at $Z^\star$.

1.8

⋆ Can the following objection be met?

> Where is the runner when he goes out of existence? He cannot be at any Z-point since, by hypothesis, there is always a Z-point beyond it, which means that he would not have gone through all the Z-points; but if he goes out of existence at or beyond $Z^\star$, then he reached $Z^\star$, and so the sufficiency claim has not been refuted.

Part of the puzzle here lies, I think, in the exact nature of the correspondence that we are setting up between mathematical series and physical space. We have two different things: on the one hand, a series of mathematical points, the Z-series, and on the other hand, a series of physical points composing the physical racetrack. A mathematical series, like the Z-series, may have no last member. In this case, it is not clear how we are to answer the question "To what physical length does this series of mathematical points correspond?" That this is a genuine question is obscured by the fact that we can properly apply the word "point" both to a mathematical abstraction and to a position in physical space. However, lengths as ordinarily thought of have *two* ends. If a length can be correlated with a mathematical series with only *one* end, like the Z-series, this can only be by stipulation. So if we are to think of part of the racetrack as a length, a two-ended length, corresponding to the mathematically defined Z-series, a one-ended length, we can but stipulate that what corresponds to the physical length is the series from Z to $Z^\star$. Given this, it is obvious that traversing the length corresponding to the Z-series is enough to get the runner to $Z^\star$. On this view, the paradox is resolved by rejecting the argument for (b), and accepting that for (a) – modified, perhaps, by the quibble about the runner continuing to exist.

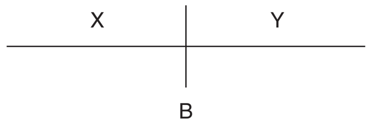

Figure 1.1

This suggestion can be strengthened by the following consideration. Suppose we divide a line into two discrete parts, X and Y, by drawing a perpendicular that cuts it at a point B, as in figure 1.1. The notions of *line, division*, and so on are to be just our ordinary ones, whatever they are, and not some mathematical specification of them. Since B is a spatial point, it must be somewhere. So is it in X or in Y or both? We cannot say that it is in both X and Y, since by hypothesis these are discrete lines; that is, they have no point in common. However, it would seem that any reason we could have for saying that B is in X is as good a reason for saying that it is in Y. So, if it is in either, then it is in both, which is impossible.

If we try to represent the intuitive idea in the diagram in mathematically precise terms, we have to make a choice. Let us think of lengths in terms of sets of (mathematical) points. If X and Y are to be discrete (have no points in common), we must choose between assigning B to X (as its last member, in a left-to-right ordering) and assigning B to Y (as its first member). If we make the first choice, then Y has no first member; if we make the second choice, then X has no last member. So far as having an adequate model for physical space goes, there seems to be nothing to determine this choice – it seems that we are free to stipulate.

Suppose we make the first choice, according to which B is in X. Imagine an archer being asked to shoot an arrow that traverses the whole of a physical space corresponding to X, without entering into any of the space corresponding to Y. There is no conceptual problem about this instruction: the arrow must be shot from the leftmost point of X and land at B. Now imagine an archer being asked to shoot an arrow that traverses the whole of a physical space corresponding to Y, without entering into any of the space corresponding to X. This time there appears to be a conceptual problem. The arrow cannot land at the point in space corresponding to B because, by stipulation, B has been allocated to X and so lies outside Y; but nor can the arrow land anywhere in Y, since for any point in Y there is one between it and B. There is no point that is the *first* point to the right of B.

What is odd about this contrast – the ease of occupying all of X and none of Y, the difficulty of occupying all of Y and none of X – is that *which* task is problematic depends upon a stipulation. If we had made the other choice, stipulating that B is to belong to Y, the difficulties would have been transposed.

Two real physical tasks, involving physical space, cannot vary in their difficulty according to some stipulation about how B is to be allocated. There is some discrepancy here between the abstract mathematical space-like notions, and our notions of physical space.

If we think of X and Y as genuine lengths, as stretches of physical space, the difficulty we face can be traced to the source already mentioned: lengths – for example, the lengths of racetracks – have *two* ends. If B belongs to X and not Y, then Y lacks a left-hand end: it cannot have B as its end, since B belongs to X and not Y (by hypothesis); but it cannot have any point to the right of B as its left end, for there will always be a Y-point to the left of any point that is to the right of B.

The difficulty comes from the assumption that the point B has partially to *compose* a line to which it belongs, so that to say it belongs to X and Y would be inconsistent with these being non-overlapping lines. For an adequate description of physical space, we need a different notion: one that allows, for example, that two distinct physical lengths, arranged like X and Y, should touch without overlapping. We need the notion of a boundary that does not itself occupy space.

If we ask what region of space – thought of in the way we think of racetracks, as having two ends – corresponds to the points on the Z-series, the only possible answer would appear to be the region from Z to $Z^\star$. This explains why the argument for sufficiency is correct, despite the point noted in the argument against it. $Z^\star$ does not belong to the Z-series, but it does belong to the region of space that corresponds to the Z-series.

In these remarks, I have assumed that we have coherent spatial notions, for example, that of (two-ended) length, and that if some mathematical structure does not fit with these notions, then so much the worse for the view that the structure gives a correct account of our spatial notions. In the circumstances, this pattern of argument is suspect, for it is open to the following Zeno-like response: the *only* way we could hope to arrive at coherent spatial notions is through these mathematical structures. If this way fails – if the mathematical structures do not yield all we want – then we are forced to admit that we were after the impossible, and that there is no way of making sense of our spatial concepts.

The upshot is that a full response to Zeno's Racetrack paradox would require a detailed elaboration and justification of our spatial concepts. This is the task Zeno set us – a task that each generation of philosophers of space and time rightly feels it must undertake anew.

1.5 Achilles and the Tortoise

We can restate this most famous of paradoxes using some Racetrack terminology. The Z-series can be redefined as follows: Z is Achilles' starting point; Z_1 is the tortoise's starting point; Z_2 is the point that the tortoise reaches while Achilles is getting to Z_1; and so on. $Z^\star$ becomes the point at which, we all believe, Achilles will catch the tortoise, and the "proof" is that Achilles, like the runner before him, will never reach $Z^\star$.

We can see this as nothing more, in essentials, than the Racetrack, but with a receding finishing line. The paradoxical claim is this: Achilles can never get to $Z^\star$ because however many points in the Z-series he has occupied, there are still more Z-points ahead before he gets to $Z^\star$. Furthermore, we cannot expect him to complete an infinity of "tasks" (moving through Z-points) in a finite time. An adequate response to the Racetrack will be easily converted into an adequate response to this version of the Achilles.

In such an interpretation of the paradox, the tortoise has barely a walk-on part to play. Let us see if we can do him more justice. One attempt is this:

> The tortoise is always ahead of Achilles if Achilles is at a point in the Z-series. But how is this consistent with the supposition that they reach $Z^\star$ at the same time? If the tortoise is always ahead in the Z-series, must he not emerge from it before Achilles?

This makes for a rather superficial paradox. It is trivial that the tortoise is ahead of Achilles all the time until Achilles has drawn level: he is ahead until $Z^\star$. Given that both of them can travel through all the Z-points, which was disputed in the Racetrack but which is not now challenged, there is no reason why they should not complete this task at the same point in space and time. So I have to report that I can find nothing of substantial interest in this paradox that has not already been discussed in connection with the Racetrack.

1.6 The Arrow

At any instant of time, the flying arrow "occupies a space equal to itself." That is, the arrow at an instant cannot be moving, for motion takes a period of time, and a temporal instant is conceived as a point, not itself having duration. It follows that the arrow is at rest at every instant, and so does not move. What goes for arrows goes for everything: nothing moves.

Aristotle gives a very brief report of this paradoxical argument, and concludes that it shows that "time is not composed of indivisible instants" (Aristotle 1970, Z9. 239b 5). This is one possible response, though one that would nowadays lack appeal. Classical mechanics purports to make sense not only of velocity at an instant but also of various more sophisticated notions: rate of change of velocity at an instant (i.e. instantaneous acceleration or deceleration), rate of change of acceleration at an instant, and so on.

Another response is to accept that the arrow is at rest at every instant, but deny that it follows that it does not move. What is required for the arrow to move, it may be said, is not that it move-at-an-instant, which is clearly an impossibility (given the semi-technical notion of *instant* in question), but rather that it be at different places at different instants. An instant is not long enough for motion to occur, for motion is a relation between an object, places, and various instants. If a response along these lines can be justified, there is no need to accept Aristotle's conclusion.

Suppose we set out Zeno's argument like this:

(1) At each instant, the arrow does not move.
(2) A stretch of time is composed of instants.
Conclusion: In any stretch of time, the arrow does not move.

Then the response under discussion is that this argument is not valid: the premises are true, but they do not entail the conclusion.

If the first premise is to be acceptable, it must be understood in a rather special way, which provides the key to the paradox. It must be understood as making a claim which does not immediately entail that the arrow is at rest. The question of whether something is moving or at rest "at an instant" is one that essentially involves other instants. An object is at rest at an instant just on condition that it is at the same place at all nearby instants; it is in motion at an instant just on condition that it is in different places at nearby instants. Nothing about the arrow and a single instant alone can fix either that it is moving then or at rest then. In short, the first premise, if acceptable, cannot be understood as saying that at each instant the arrow is at rest.

Once the first premise is properly understood, it is easy to see why the argument is fallacious. The conclusion that the arrow is always at rest says of each instant that the arrow is in the same place at neighboring instants. No such information is contained in the premises. If we think it is implicit in the premises, this is probably because we are failing to distinguish between the claim – interpretable as true – that at each instant the arrow does not move, and the false claim that it is *at rest* at each instant.

If this is correct, then the Arrow paradox is an example of one in which the unacceptable conclusion (nothing moves) comes from an acceptable premise (no motion occurs "during" an instant) by unacceptable reasoning.

Suggested reading

A good starting point is Nick Huggett's *Stanford Encyclopedia* article, available at: plato.stanford.edu/entries/paradox-zeno/.

Salmon (1970) contains the articles by Thomson (1954) and Benacerraf (1962) from which I drew the discussion of infinity machines, as well as many other important articles, including a clear introductory survey by Salmon. It also has an excellent bibliography. For a fine introduction to the philosophy of space and time, including a chapter on Zeno's paradoxes, see Salmon (1980).

For a historical account see Vlastos (1967). For an advanced discussion, see Grünbaum (1967).

The quotation from Peirce, written late in his life, is not representative. In many other places, he discusses Zeno's paradoxes very seriously. However, it is not uncommon for people to see a paradox as trivial once they think they have a definitive solution to it. The cure for this reaction is to try to persuade someone else of one's "solution."

The phrase "medically impossible" comes from Russell (1936, p. 143).